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# Continuous wavelet transforms and non-commutative Fourier analysis

By

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## Abstract

We discuss continuous wavelet transforms for the semidirect product group of a unimodular (not necessarily commutative) normal subgroup  $N$  with a closed subgroup  $H$  of  $\text{Aut}(N)$ , which is a generalization of the wavelet theory for an affine transformation group on a vector space. The operator-valued Fourier transform for  $N$  plays a substantial role in the arguments.

## § 1. Introduction

Let  $G$  be a locally compact group, and  $(\pi, \mathcal{H})$  an irreducible unitary representation of  $G$ . The representation  $\pi$  is said to be *square-integrable* if there exists a vector  $\phi \in \mathcal{H}$  for which  $\int_G |(\pi(g)\phi|\phi)_{\mathcal{H}}|^2 dg < +\infty$ , where  $dg$  is a left Haar measure on  $G$ . Such  $\phi$  is called an *admissible vector* of  $\pi$ . If  $(\pi, \mathcal{H})$  is square-integrable, there exists a unique positive self-adjoint operator (the *Duflo-Moore operator*)  $C_{\pi}$  with the following two properties ([7], [13]):

(C1)  $\phi$  is admissible  $\Leftrightarrow \phi \in \text{dom}(C_{\pi})$ ,

(C2) For  $f_1, f_2 \in \mathcal{H}$  and  $\phi_1, \phi_2 \in \text{dom}(C_{\pi})$ , one has

$$\int_G (f_1|\pi(g)\phi_1)_{\mathcal{H}} (\pi(g)\phi_2|f_2)_{\mathcal{H}} dg = (f_1|f_2)_{\mathcal{H}} (C_{\pi}\phi_1|C_{\pi}\phi_2)_{\mathcal{H}}.$$

When  $G$  is unimodular,  $C_{\pi}$  is a scalar operator. Furthermore, if  $G$  is a compact group, then  $C_{\pi} = (\dim \mathcal{H})^{-1/2} \text{Id}$ . Thus,  $C_{\pi}^{-2}$  is called *the formal degree* of  $\pi$  in general.

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Regarding the equality in (C2) as an identity for  $f_2 \in \mathcal{H}$ , we obtain

$$(1.1) \quad f_1 = \frac{1}{(C_\pi \phi_1 | C_\pi \phi_2)_\mathcal{H}} \int_G (f_1 | \pi(g) \phi_1)_\mathcal{H} \pi(g) \phi_2 \, dg,$$

where the integral is taken in the weak sense. For an admissible vector  $\phi \in \text{dom}(C_\pi)$ , the continuous wavelet transform  $W_\phi$  is defined as a linear map from the Hilbert space  $\mathcal{H}$  into the space  $C(G)$  of continuous functions on the group  $G$  defined by

$$W_\phi f(g) := (f | \pi(g) \phi)_\mathcal{H} \quad (f \in \mathcal{H}, \, g \in G).$$

Then (1.1) tells us that the inverse formula of  $W_\phi$  is given by

$$(1.2) \quad f = \frac{1}{c_\phi} \int_G W_\phi f(g) \pi(g) \phi \, dg,$$

where  $c_\phi = \|C_\pi \phi\|_\mathcal{H}^2$ .

For instance, let us consider the affine transformation group  $G_{\text{aff}}$  on the real line  $\mathbb{R}$  consisting of the maps  $g_{b,a} : \mathbb{R} \ni x \mapsto ax + b \in \mathbb{R}$  with  $a \in \mathbb{R} \setminus \{0\}$  and  $b \in \mathbb{R}$ . The unitary representation  $L$  of  $G_{\text{aff}}$  is defined on the Hilbert space  $L^2(\mathbb{R})$  by  $L(g_{b,a})f(x) := |a|^{-1/2} f(g_{b,a}^{-1}x)$  ( $x \in \mathbb{R}$ ). This representation is square-integrable, and the Duflo-Moore operator is given by

$$(C_L \phi)^\wedge(\xi) = \sqrt{\frac{2\pi}{|\xi|}} \hat{\phi}(\xi) \quad (\xi \in \mathbb{R}),$$

where  $\hat{\phantom{x}}$  stands for the Fourier transform given by  $\hat{\phi}(\xi) := \int_{\mathbb{R}} e^{ix\xi} \phi(x) \, dx$  ( $\xi \in \mathbb{R}$ ). Then  $c_\phi = \|C_L \phi\|^2$  equals  $\int_{\mathbb{R} \setminus \{0\}} |\hat{\phi}(\xi)|^2 |\xi|^{-1} \, d\xi$ , and if  $c_\phi < +\infty$ , the equality (1.2) yields the Calderón formula

$$(1.3) \quad f = \frac{1}{c_\phi} \int_{\mathbb{R}} \int_{\mathbb{R}^\times} W_\phi f(b, a) L(g_{b,a}) \phi \frac{db \, da}{|a|^2} \quad (f \in L^2(\mathbb{R})),$$

where  $W_\phi f(b, a) = W_\phi f(g_{b,a}) = |a|^{-1/2} \int_{\mathbb{R}} f(x) \overline{\phi(\frac{x-b}{a})} \, dx$ .

The results about the wavelet transform for  $G_{\text{aff}}$  have been generalized to a multi-dimensional affine group  $G = H \rtimes \mathbb{R}^n$ , where  $H$  is a closed subgroup of  $GL(\mathbb{R}^n)$  (see [4], [8] and [10] for example). In this article, we consider a further generalization to the case that  $G$  is a semidirect product group  $N \rtimes H$ , where  $N$  is a unimodular (not necessarily commutative) group, and  $H$  is a closed subgroup of  $\text{Aut}(N)$  satisfying certain conditions. Although the situation becomes complicated if  $N$  is not commutative, the argument goes in parallel with the commutative case, where the operator-valued Fourier transform for  $N$  plays a substantial role instead of the ordinary Fourier transform.

The content of Sections 2–4 is essentially a summary of [16], while Section 5 is devoted to discuss a concrete example that  $N$  is the Heisenberg group. Such Heisenberg

case is already studied by He-Liu [14], whereas we shall present a new example of admissible vector  $\phi_{\pm, \alpha} \in L^2(N)$ . In this article, we write  $\mathbb{T}$  for the set of complex numbers with absolute value 1. For a Hilbert space  $\mathcal{H}$ , we denote by  $\mathcal{B}(\mathcal{H})$ , (resp.  $\mathcal{B}_{\text{HS}}(\mathcal{H})$ ,  $\mathcal{B}_{\text{Tr}}(\mathcal{H})$ ,  $U(\mathcal{H})$ ) the space of bounded (resp. Hilbert-Schmidt, trace class, unitary) operators on  $\mathcal{H}$ .

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## § 2. Preliminaries

Let  $N$  be a separable locally compact unimodular group of type I. We denote by  $\hat{N}$  the unitary dual of  $N$ , that is, the set of equivalence classes of irreducible unitary representations of  $N$ . For each  $\lambda \in \hat{N}$ , we take a unitary representation  $(\pi_\lambda, \mathcal{H}_\lambda)$  for which  $\lambda$  equals the equivalence class  $[\pi_\lambda]$  of  $\pi_\lambda$ . Let us fix a Haar measure  $\nu$  on  $N$ . For  $f \in L^1(N)$ , we define the bounded operator  $\pi_\lambda(f) \in \mathcal{B}(\mathcal{H}_\lambda)$  by  $\pi_\lambda(f) := \int_N f(n) \pi_\lambda(n) d\nu(n)$ . It is known that the Plancherel measure  $\mu$  on  $\hat{N}$  is uniquely determined by the abstract Plancherel formula [6]:

$$(2.1) \quad \int_N |f(n)|^2 d\nu(n) = \int_{\hat{N}} \|\pi_\lambda(f)\|_{\text{HS}}^2 d\mu(\lambda) \quad (f \in L^1(N) \cap L^2(N)),$$

where  $\|\cdot\|_{\text{HS}}$  stands for the Hilbert-Schmidt norm. We define the operator-valued Fourier transform  $\mathbf{F} : L^2(N) \rightarrow \int_{\hat{N}}^{\oplus} \mathcal{B}_{\text{HS}}(\mathcal{H}_\lambda) d\mu(\lambda)$  as the unitary isomorphism which is the extension of the map  $L^1(N) \cap L^2(N) \ni f \mapsto (\pi_\lambda(f))_{\lambda \in \hat{N}} \in \int_{\hat{N}}^{\oplus} \mathcal{B}_{\text{HS}}(\mathcal{H}_\lambda) d\mu(\lambda)$ . The inverse formula of  $\mathbf{F}$  is given as follows:

**Proposition 2.1** ([11, Theorem 4.15]). *Let  $(A(\lambda))_{\lambda \in \hat{N}}$  be an element of the direct integral  $\int_{\hat{N}}^{\oplus} \mathcal{B}_{\text{Tr}}(\mathcal{H}_\lambda) d\mu(\lambda)$  of the Banach space  $\mathcal{B}_{\text{Tr}}(\mathcal{H}_\lambda)$ . Define a function  $f$  on  $N$  by*

$$f(n) := \int_{\hat{N}} \text{tr } A(\lambda) \pi_\lambda(n)^* d\mu(\lambda) \quad (n \in N).$$

*Then  $f$  belongs to  $L^2(N)$  if and only if  $(A(\lambda))_{\lambda \in \hat{N}} \in \int_{\hat{N}}^{\oplus} \mathcal{B}_{\text{HS}}(\mathcal{H}_\lambda) d\mu(\lambda)$ . In that case, one has  $\mathbf{F}f(\lambda) = A(\lambda)$  (a.a.  $\lambda \in \hat{N}$ ).*

Let  $H$  be a closed subgroup of the automorphism group  $\text{Aut}(N)$  on  $N$ , and  $G$  the semidirect product group  $N \rtimes H$ . We write the action of  $h$  to  $n \in N$  as  $h \cdot n$ . For  $h \in H$ , we have a positive number  $\delta(h)$  for which  $d\nu(h \cdot n) = \delta(h) d\nu(n)$  ( $n \in N$ ). Clearly,  $\delta : H \rightarrow \mathbb{R}_+$  is a representation of  $H$ . Define the unitary representation  $L$  of  $G$  on  $L^2(N)$  by

$$(2.2) \quad \begin{aligned} L(h)f(n_0) &:= \delta(h)^{-1/2} f(h^{-1} \cdot n_0), \\ L(n)f(n_0) &:= f(n^{-1}n_0) \quad (f \in L^2(N), h \in H, n, n_0 \in N). \end{aligned}$$

It is easy to see that the representation  $L$  is equivalent to the induced representation  $\text{Ind}_H^G \mathbf{1}$ , where  $\mathbf{1}$  is the trivial representation of  $H$ .

We define the action of  $H$  on the unitary dual  $\hat{N}$  by  $h \cdot \lambda := [\pi_\lambda \circ h^{-1}]$  ( $h \in H$ ,  $\lambda \in \hat{N}$ ). Then we have a unitary operator  $C(h, \lambda) : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{h \cdot \lambda}$  with the property

$$C(h, \lambda) \pi_\lambda(h^{-1} \cdot n) = \pi_{h \cdot \lambda}(n) C(h, \lambda) \quad (n \in N).$$

The operator  $C(h, \lambda)$  is unique up to multiple by elements of  $\mathbb{T}$  by Schur's lemma. Moreover, for  $h, h' \in H$  and  $\lambda \in \hat{N}$ , we have

$$(2.3) \quad C(hh', \lambda) = s_{h, h', \lambda} C(h, h' \cdot \lambda) C(h', \lambda)$$

with  $s_{h, h', \lambda} \in \mathbb{T}$ . Thus we have a uniquely determined operator  $D(h, \lambda) : \mathcal{B}(\mathcal{H}_\lambda) \ni T \mapsto C(h, \lambda) T C(h, \lambda)^* \in \mathcal{B}(\mathcal{H}_{h \cdot \lambda})$ , which satisfies the chain rule

$$D(hh', \lambda) = D(h, h' \cdot \lambda) D(h', \lambda).$$

Using the operator-valued Fourier transform  $\mathbf{F}$ , we describe the representation  $(L, L^2(N))$  of  $G$  as follows:

**Proposition 2.2.** *For  $f \in L^2(N)$ ,  $h \in H$  and  $n \in N$ , one has*

$$(2.4) \quad \mathbf{F}[L(h)f](\lambda) = \delta(h)^{1/2} D(h, h^{-1} \cdot \lambda) \mathbf{F}f(h^{-1} \cdot \lambda),$$

$$(2.5) \quad \mathbf{F}[L(n)f](\lambda) = \pi_\lambda(n) \mathbf{F}f(\lambda)$$

for almost all  $\lambda \in \hat{N}$  with respect to the Plancherel measure  $\mu$ .

### § 3. Decomposition of $L^2(N)$

For  $\lambda \in \hat{N}$ , let  $\mathcal{O}_\lambda^* \subset \hat{N}$  be the  $H$ -orbit through  $\lambda$ . Now let us assume that there exist elements  $\lambda_1, \dots, \lambda_K \in \hat{N}$  satisfying the following conditions:

(H1)  $\mu(\mathcal{O}_{\lambda_k}^*) > 0$  ( $k = 1, \dots, K$ ),

(H2) The stabilizer  $H_k := \{h \in H; h \cdot \lambda_k = \lambda_k\}$  is compact for all  $k = 1, \dots, K$ .

(H3) The map  $H/H_k \ni hH_k \mapsto h \cdot \lambda_k \in \mathcal{O}_{\lambda_k}^*$  is a homeomorphism for  $k = 1, \dots, K$ , where the topology on  $\mathcal{O}_{\lambda_k}^*$  is induced from the Fell topology on  $\hat{N}$ .

(H4)  $\mathcal{O}_{\lambda_k}^* \cap \mathcal{O}_{\lambda_l}^* = \emptyset$  ( $k \neq l$ ) and  $\mu(\hat{N} \setminus \bigsqcup_{k=1}^K \mathcal{O}_{\lambda_k}^*) = 0$ .

Under this assumption, we shall see in Sections 3 and 4 that the unitary representation  $(L, L^2(N))$  of  $G$  is decomposed into the direct sum of countably many irreducible subrepresentations, and such subrepresentations are all square-integrable. Their Duflo-Moore operators are described by using the operator-valued Fourier transform  $\mathbf{F}$ .

Thanks to (2.3), we have a projective unitary representation  $\tau_k : H_k \ni h \mapsto C(h, \lambda_k) \in U(\mathcal{H}_{\lambda_k})$  of the group  $H_k$  for  $k = 1, \dots, K$ . Since  $H_k$  is compact, we have an

irreducible decomposition  $\mathcal{H}_{\lambda_k} = \sum_{\alpha \in A_k}^{\oplus} \mathcal{H}_{\lambda_k, \alpha}$ , where  $A_k$  is an at most countable index set. The subspaces  $\mathcal{H}_{\lambda_k, \alpha}$  are finite dimensional. For  $\lambda = \tilde{h} \cdot \lambda_k \in \mathcal{O}_{\lambda_k}^*$  with  $\tilde{h} \in H$ , we put  $\mathcal{H}_{\lambda, \alpha} := C(\tilde{h}, \lambda_k) \mathcal{H}_{\lambda_k, \alpha}$  ( $\alpha \in A_k$ ), where the right-hand side is independent of the choice of  $\tilde{h}$ . Moreover we have  $\mathcal{H}_{\lambda} = \sum_{\alpha \in A_k}^{\oplus} \mathcal{H}_{\lambda, \alpha}$ , which gives an irreducible decomposition of the projective representation  $\tau_{\lambda} : H_{\lambda} \ni h \mapsto C(h, \lambda) \in U(\mathcal{H}_{\lambda})$  of the compact group  $H_{\lambda} := \{h \in H; h \cdot \lambda = \lambda\}$ . On the other hand, the relation

$$(3.1) \quad C(h, \lambda) \mathcal{H}_{\lambda, \alpha} = \mathcal{H}_{h \cdot \lambda, \alpha}$$

holds for  $h \in H$ ,  $\lambda \in \mathcal{O}_{\lambda_k}^*$  and  $\alpha \in A_k$  in general. Using the orthogonal projection  $P_{\lambda, \alpha} : \mathcal{H}_{\lambda} \rightarrow \mathcal{H}_{\lambda, \alpha}$ , we define

$$(3.2) \quad \mathcal{B}_{\lambda, \alpha} := \{T \in \mathcal{B}_{\text{HS}}(\mathcal{H}_{\lambda}); TP_{\lambda, \alpha} = T\}.$$

If we identify  $\mathcal{B}_{\text{HS}}(\mathcal{H}_{\lambda})$  with the tensor product  $\mathcal{H}_{\lambda} \otimes \overline{\mathcal{H}_{\lambda}}$ , the space  $\mathcal{B}_{\lambda, \alpha}$  is nothing but  $\mathcal{H}_{\lambda} \otimes \overline{\mathcal{H}_{\lambda, \alpha}}$ . Thus we see that

$$(3.3) \quad \mathcal{B}_{\text{HS}}(\mathcal{H}_{\lambda}) = \sum_{\alpha \in A_k}^{\oplus} \mathcal{B}_{\lambda, \alpha},$$

while (3.1) yields

$$(3.4) \quad D(h, \lambda) \mathcal{B}_{\lambda, \alpha} = \mathcal{B}_{h \cdot \lambda, \alpha}.$$

Now we set

$$(3.5) \quad \begin{aligned} L_{k, \alpha}(N) &:= \mathbf{F}^{-1} \left( \int_{\mathcal{O}_{\lambda_k}^*}^{\oplus} \mathcal{B}_{\lambda, \alpha} d\mu(\lambda) \right) \\ &= \left\{ f \in L^2(N); \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda) P_{\lambda, \alpha} & (\text{a.a. } \lambda \in \mathcal{O}_{\lambda_k}^*) \\ \mathbf{F}f(\lambda) = 0 & (\text{otherwise}) \end{array} \right\} \end{aligned}$$

for  $k = 1, \dots, K$  and  $\alpha \in A_k$ . By (3.3) and (H4), we have

$$(3.6) \quad L^2(N) = \sum_{1 \leq k \leq K}^{\oplus} \sum_{\alpha \in A_k}^{\oplus} L_{k, \alpha}(N),$$

and each  $L_{k, \alpha}(N)$  is  $G$ -invariant thanks to Proposition 2.2 and (3.4).

#### § 4. Main results

Fixing a left Haar measure  $d_H$  on  $H$ , we define a left Haar measure  $d_G$  on  $G$  by  $d_G(nh) := \delta(h)^{-1} d\nu(n) d_H(h)$  ( $n \in N$ ,  $h \in H$ ). Let us consider the square-integrability

of the matrix coefficients  $(f|L(g)\phi)$  ( $f, \phi \in L_{k,\alpha}(N)$ ,  $g \in G$ ) of the representation  $(L, L_{k,\alpha}(N))$  of  $G$ . By (2.1) and (2.5), we have

$$(4.1) \quad \begin{aligned} (f|L(n)L(h)\phi) &= \int_{\mathcal{O}_{\lambda_k}^*} \operatorname{tr} \mathbf{F}f(\lambda) \mathbf{F}[L(n)L(h)\phi](\lambda)^* d\mu(\lambda) \\ &= \int_{\mathcal{O}_{\lambda_k}^*} \operatorname{tr} (\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*) \pi_\lambda(n)^* d\mu(\lambda). \end{aligned}$$

We note that the operator  $\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*$  is of trace class because both  $\mathbf{F}f(\lambda)$  and  $\mathbf{F}[L(h)\phi](\lambda)$  are Hilbert-Schmidt operators. Furthermore,  $(\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*)_{\lambda \in \mathcal{O}_{\lambda_k}^*}$  belongs to  $\int_{\mathcal{O}_{\lambda_k}^*}^{\oplus} \mathcal{B}_{\operatorname{Tr}}(\mathcal{H}_\lambda) d\mu(\lambda)$  because

$$(4.2) \quad \begin{aligned} &\int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*\|_{\operatorname{Tr}} d\mu(\lambda) \\ &\leq \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda)\|_{\operatorname{HS}} \|\mathbf{F}[L(h)\phi](\lambda)\|_{\operatorname{HS}} d\mu(\lambda) \\ &\leq \left\{ \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda)\|_{\operatorname{HS}}^2 d\mu(\lambda) \right\}^{1/2} \left\{ \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}[L(h)\phi](\lambda)\|_{\operatorname{HS}}^2 d\mu(\lambda) \right\}^{1/2} \\ &= \|f\| \|L(h)\phi\| = \|f\| \|\phi\| < +\infty, \end{aligned}$$

where  $\|\cdot\|_{\operatorname{Tr}}$  denotes the Trace norm. Now we assume that

$$\int_G |(f|L(g)\phi)|^2 d_G(g) = \int_H \int_N |(f|L(n)L(h)\phi)|^2 \delta(h)^{-1} d\nu(n) d_H(h) < +\infty.$$

Then  $\int_N |(f|L(n)L(h)\phi)|^2 d\nu(n)$  is finite for almost all  $h \in H$ . Thus, applying Proposition 2.1, we see from (4.1) and (2.1) that

$$(4.3) \quad \int_N |(f|L(n)L(h)\phi)|^2 d\nu(n) = \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*\|_{\operatorname{HS}}^2 d\mu(\lambda) \quad (\text{a.a. } h \in H).$$

Therefore the integral  $\int_G |(f|L(g)\phi)|^2 d_G(g)$  is equal to

$$(4.4) \quad \int_H \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*\|_{\operatorname{HS}}^2 \delta(h)^{-1} d\mu(\lambda) d_H(h).$$

Now we observe that

$$(4.5) \quad \begin{aligned} &\int_H \|\mathbf{F}f(\lambda) \mathbf{F}[L(h)\phi](\lambda)^*\|_{\operatorname{HS}}^2 \delta(h)^{-1} d_H(h) \\ &= \int_H \int_{H_\lambda} \left( \operatorname{tr} \mathbf{F}f(\lambda) \mathbf{F}[L(h_1 h)\phi](\lambda)^* \mathbf{F}[L(h_1 h)\phi](\lambda) \mathbf{F}f(\lambda)^* \right) \delta(h_1 h)^{-1} dh_1 d_H(h), \end{aligned}$$

where  $dh_1$  is the normalized Haar measure on the compact group  $H_\lambda$ . Since  $L(h_1h)\phi$  belongs to  $L_{k,\alpha}(N)$ , we have by (3.5)

$$P_{\lambda,\alpha} \mathbf{F}[L(h_1h)\phi](\lambda)^* \mathbf{F}[L(h_1h)\phi](\lambda) P_{\lambda,\alpha} = \mathbf{F}[L(h_1h)\phi](\lambda)^* \mathbf{F}[L(h_1h)\phi](\lambda),$$

which means that we can regard  $\mathbf{F}[L(h_1h)\phi](\lambda)^* \mathbf{F}[L(h_1h)\phi](\lambda)$  as a linear operator on the finite dimensional vector space  $\mathcal{H}_{\lambda,\alpha}$ . Furthermore, applying Schur's lemma to the representation  $(\tau_\lambda, \mathcal{H}_{\lambda,\alpha})$  of  $H_\lambda$ , we get

$$\begin{aligned} & \int_{H_\lambda} \delta(h_1h)^{-1} \mathbf{F}[L(h_1h)\phi](\lambda)^* \mathbf{F}[L(h_1h)\phi](\lambda) dh_1 \\ &= (\dim \mathcal{H}_{\lambda,\alpha})^{-1} \delta(h)^{-1} \|\mathbf{F}[L(h)\phi](\lambda)\|_{\text{HS}}^2 P_{\lambda,\alpha} \in \text{End}(\mathcal{H}_{\lambda,\alpha}), \end{aligned}$$

see [16, pp. 43–44] for the detail. Note that  $\dim \mathcal{H}_{\lambda,\alpha} = \dim \mathcal{H}_{\lambda_k,\alpha}$ , which we denote by  $n_{k,\alpha}$  in what follows. By (4.5) and (3.5), the integral (4.4) equals

$$\begin{aligned} & \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \int_H \left( \text{tr } \mathbf{F}f(\lambda) P_{\lambda,\alpha} \mathbf{F}f(\lambda)^* \right) \|\mathbf{F}[L(h)\phi](\lambda)\|_{\text{HS}}^2 \delta(h)^{-1} d_H(h) d\mu(\lambda) \\ &= \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \int_H \|\mathbf{F}f(\lambda)\|_{\text{HS}}^2 \|\mathbf{F}[L(h)\phi](\lambda)\|_{\text{HS}}^2 \delta(h)^{-1} d_H(h) d\mu(\lambda), \end{aligned}$$

and the right-hand side is rewritten as

$$\frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda)\|_{\text{HS}}^2 \left( \int_H \|\mathbf{F}\phi(h^{-1} \cdot \lambda)\|_{\text{HS}}^2 d_H(h) \right) d\mu(\lambda)$$

by (2.4). Thanks to [16, Proposition 3], the integral  $\int_H \|\mathbf{F}\phi(h^{-1} \cdot \lambda)\|_{\text{HS}}^2 d_H(h)$  does not depend on  $\lambda \in \mathcal{O}_{\lambda_k}^*$ , and it is equal to  $\int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}\phi(\lambda)\|_{\text{HS}}^2 D_k(\lambda) d\mu(\lambda)$  with a certain  $H$ -relatively invariant function  $D_k$  on  $\mathcal{O}_{\lambda_k}^*$ . Therefore we have

$$\begin{aligned} +\infty &> \int_G |(f|L(g)\phi)|^2 d_G(g) = \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}f(\lambda)\|_{\text{HS}}^2 d\mu(\lambda) \cdot \frac{1}{n_{k,\alpha}} \int_H \|\mathbf{F}\phi(h^{-1} \cdot \lambda)\|_{\text{HS}}^2 d_H(h) \\ &= \|f\|^2 \cdot \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}\phi(\lambda)\|_{\text{HS}}^2 D_k(\lambda) d\mu(\lambda). \end{aligned}$$

In particular, if  $f \neq 0$ , then

$$(4.6) \quad c(\phi) := \frac{1}{n_{k,\alpha}} \int_{\mathcal{O}_{\lambda_k}^*} \|\mathbf{F}\phi(\lambda)\|_{\text{HS}}^2 D_k(\lambda) d\mu(\lambda) < +\infty$$

and

$$(4.7) \quad \int_G |(f|L(g)\phi)|^2 d_G(g) = c(\phi) \|f\|^2.$$



Conversely, if  $\phi$  satisfies the condition (4.6), the integral (4.4) converges for any  $f \in L_{k,\alpha}(N)$ , so that the right-hand side of (4.3) converges for almost all  $h \in H$ . Therefore Proposition 2.1 implies (4.3), so that we get (4.7) again and thus,  $\int_G |(f|L(g)\phi)|^2 d_G(g) < +\infty$ .

**Theorem 4.1.** *The unitary representation  $(L, L_{k,\alpha}(N))$  of  $G$  is irreducible.*

*Proof.* Let  $\mathcal{L}$  be a nonzero invariant subspace of  $L_{k,\alpha}(N)$ . The orthogonal complement  $\mathcal{L}^\perp \subset L_{k,\alpha}(N)$  is also invariant. We take  $\phi \in \mathcal{L} \setminus \{0\}$  and  $f \in \mathcal{L}^\perp$ . By (4.7) we have

$$0 = \int_G |(f|L(g)\phi)|^2 d_G(g) = c(\phi)\|f\|^2,$$

which implies  $f = 0$ . Actually, the argument is valid even if  $c(\phi)$  is not finite. Therefore  $\mathcal{L}^\perp = \{0\}$  and Theorem 4.1 is proved.  $\square$

Furthermore, we deduce from (4.6) and (4.7) the following result.

**Theorem 4.2.** *The representation  $(L, L_{k,\alpha}(N))$  is square-integrable, whose Dufluo-Moore operator  $C_{k,\alpha}$  is given by*

$$\mathbf{F}[C_{k,\alpha}\phi](\lambda) = \sqrt{\frac{D_k(\lambda)}{n_{k,\alpha}}} \mathbf{F}\phi(\lambda) \quad (\phi \in L_{k,\alpha}(N), \lambda \in \mathcal{O}_{\lambda_k}^*).$$

A classification of the representations  $(L, L_{k,\alpha}(N))$  is also given in [16]. The result is as follows:

**Theorem 4.3** ([16, Theorem 4]). *The unitary representations  $(L, L_{k,\alpha}(N))$  and  $(L, L_{k',\alpha'}(N))$  of  $G$  are equivalent if and only if  $k = k'$  and the projective representations  $(\tau_k, \mathcal{H}_{k,\alpha})$  and  $(\tau_k, \mathcal{H}_{k,\alpha'})$  of  $H_k$  are equivalent, that is, there exists an isometry  $A : \mathcal{H}_{k,\alpha} \rightarrow \mathcal{H}_{k,\alpha'}$  such that  $\tau_k(h) \circ A = A \circ \tau_k(h)$  for all  $h \in H_k$ .*

Keeping the decomposition (3.6) in mind, and applying (1.2) to each representation  $(L, L_{k,\alpha}(N))$ , we obtain

**Theorem 4.4.** *For each  $k = 1, \dots, K$  and  $\alpha \in A_k$ , take admissible vectors  $\phi_{k,\alpha} \in \text{dom}(C_{k,\alpha}) \subset L_{k,\alpha}(N)$ . Then for all  $f \in L^2(N)$  one has*

$$f = \sum_{k=1}^K \sum_{\alpha \in A_k} \frac{1}{\|C_{k,\alpha}\phi_{k,\alpha}\|^2} \int_G W_{\phi_{k,\alpha}} f(g) L(g)\phi_{k,\alpha} d_G(g),$$

where  $W_{\phi_{k,\alpha}} f(g) = (f|L(g)\phi_{k,\alpha})$ .

### § 5. Example

As an illustrative example, we shall consider the case that the unimodular group  $N$  is the Heisenberg group of  $(2\ell + 1)$ -dimension and  $H$  is isomorphic to  $\mathbb{R}_+ \times U(\ell)$ . The continuous wavelet transform for this case is first considered by He-Liu [14].

Let  $N$  be the Lie group consisting of elements  $n(z, c)$  ( $z \in \mathbb{C}^\ell$ ,  $c \in \mathbb{R}$ ) with multiplication rule

$$n(z, c)n(z', c') := (z + z', c + c' + \Im(z|z')) \quad (z, z' \in \mathbb{C}^\ell, c, c' \in \mathbb{R}),$$

where  $(z|z') := \sum_{k=1}^{\ell} z_k \bar{z}'_k$ . Define a Haar measure  $d\nu$  on  $N$  by  $d\nu(n(z, c)) := dm(z)dc$ , where  $dm$  is the standard Euclidean measure on  $\mathbb{C}^\ell$ .

For  $\zeta \in \mathbb{C}^\ell$ , we define the unitary character  $\chi_\zeta$  by  $\chi_\zeta(n(z, c)) := e^{i\Re(z|\zeta)}$ . For  $\lambda > 0$ , we define

$$\mathcal{H}_\lambda := \left\{ \varphi : \mathbb{C}^\ell \rightarrow \mathbb{C} \text{ (holomorphic)}; \|\varphi\|^2 = \frac{\lambda^\ell}{\pi^\ell} \int_{\mathbb{C}^\ell} |\varphi(w)|^2 e^{-\lambda|w|^2} dm(w) < +\infty \right\},$$

and  $\mathcal{H}_{-\lambda} := \{\bar{\varphi}; \varphi \in \mathcal{H}_\lambda\}$ . These  $\mathcal{H}_{\pm\lambda}$  are Hilbert spaces, on which we define the irreducible unitary representations  $\pi_{\pm\lambda}$  of  $N$  by

$$\begin{aligned} \pi_\lambda(n(z, c))\varphi(w) &:= e^{-i\lambda c + \lambda(w|z) - \lambda|z|^2/2} \varphi(w - z) & (\varphi \in \mathcal{H}_\lambda), \\ \pi_{-\lambda}(n(z, c))\varphi(w) &:= e^{i\lambda c + \lambda(z|w) - \lambda|z|^2/2} \varphi(w - z) & (\varphi \in \mathcal{H}_{-\lambda}). \end{aligned}$$

The Stone-von Neumann theorem states that every irreducible unitary representation of  $N$  is equivalent to one of  $\chi_\zeta$  ( $\zeta \in \mathbb{C}^\ell$ ) and  $\pi_\lambda$  ( $\lambda \in \mathbb{R} \setminus \{0\}$ ), so that  $\hat{N}$  can be identified with  $\mathbb{C}^\ell \sqcup (\mathbb{R} \setminus \{0\})$ . For  $f_1, f_2 \in L^1(N) \cap L^2(N)$ , we have

$$(f_1|f_2) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_{\mathbb{R} \setminus \{0\}} (\pi_\lambda(f_1)|\pi_\lambda(f_2))_{\text{HS}} |\lambda|^\ell d\lambda$$

by [9, Chapter I, section 5], which implies that the Plancherel measure  $\mu$  on  $\hat{N}$  is given by  $\mu(\mathbb{C}^\ell) = 0$  and  $d\mu(\lambda) = 2^{\ell-1} \pi^{-\ell-1} |\lambda|^\ell d\lambda$  ( $\lambda \in \mathbb{R} \setminus \{0\}$ ).

For  $a > 0$  and  $u \in U(\ell)$ , define  $h(a, u) \in \text{Aut}(N)$  by  $h(a, u) \cdot n(z, c) := n(auz, a^2c)$ . Then we have

$$h(a, u)h(a', u') = h(aa', uu') \quad (a, a' > 0, u, u' \in U(\ell)),$$

so that  $H := \{h(a, u); a > 0, u \in U(\ell)\}$  is a subgroup of  $\text{Aut}(N)$ . For  $h = h(a, u) \in H$ , we have  $\delta(h) = a^{2\ell+2}$ , and we define the representation  $(L, L^2(N))$  of  $G = N \rtimes H$  by (2.2). The action of  $H$  on  $\hat{N}$  is described as

$$\begin{aligned} h(a, u) \cdot \zeta &= a^{-1}u\zeta & (\zeta \in \mathbb{C}^\ell), \\ h(a, u) \cdot \lambda &= a^{-2}\lambda & (\lambda \in \mathbb{R} \setminus \{0\}). \end{aligned}$$

In particular, the intertwining operator  $C(h, \lambda) : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{a^{-2}\lambda}$  is given by

$$C(h, \lambda)\varphi(w) := \varphi(a^{-1}u^{-1}w) \quad (\varphi \in \mathcal{H}_\lambda).$$

We denote by  $\mathcal{O}_\pm^*$  the  $H$ -orbit through  $\pm 1 \in \mathbb{R} \setminus \{0\} \subset \hat{N}$ . Then  $\mathcal{O}_\pm^* = \{\lambda; \pm\lambda > 0\}$ , and the conditions (H1)–(H4) are satisfied for the two orbits. In particular, the stabilizer  $H_\lambda$  at any  $\lambda \in \mathcal{O}_+^* \sqcup \mathcal{O}_-^*$  equals the compact group  $\{h(1, u); u \in U(\ell)\} \simeq U(\ell)$ . For a non-negative integer  $\alpha \in \mathbb{Z}_{\geq 0}$ , let  $\mathcal{P}_\alpha(\mathbb{C}^\ell)$  be the space of holomorphic polynomials of degree  $\alpha$  on  $\mathbb{C}^\ell$ , and  $\overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell)$  the space  $\{\overline{\varphi}; \varphi \in \mathcal{P}_\alpha(\mathbb{C}^\ell)\}$ . Then we have for  $\lambda > 0$

$$\mathcal{H}_\lambda = \sum_{\alpha=0}^{\infty} \oplus \mathcal{P}_\alpha(\mathbb{C}^\ell), \quad \mathcal{H}_{-\lambda} = \sum_{\alpha=0}^{\infty} \oplus \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell),$$

which give the irreducible decomposition of the representation  $(\tau_{\pm\lambda}, \mathcal{H}_{\pm\lambda})$  of  $H_{\pm\lambda}$  respectively. Let  $P_{\lambda, \alpha} : \mathcal{H}_\lambda \rightarrow \mathcal{P}_\alpha(\mathbb{C}^\ell)$  and  $P_{-\lambda, \alpha} : \mathcal{H}_{-\lambda} \rightarrow \overline{\mathcal{P}}_\alpha(\mathbb{C}^\ell)$  be the orthogonal projections, and set

$$L_{\pm, \alpha}(N) := \left\{ f \in L^2(N); \begin{array}{ll} \mathbf{F}f(\lambda) = \mathbf{F}f(\lambda)P_{\lambda, \alpha} & (\text{a.a. } \lambda \in \mathcal{O}_\pm^*) \\ \mathbf{F}f(\lambda) = 0 & (\text{otherwise}) \end{array} \right\}.$$

Then an irreducible decomposition of the unitary representation  $(L, L^2(N))$  of  $G$  is given by

$$(5.1) \quad L^2(N) = \sum_{\alpha=0}^{\infty} \oplus L_{+, \alpha}(N) \oplus \sum_{\alpha=0}^{\infty} \oplus L_{-, \alpha}(N).$$

The decomposition is multiplicity-free thanks to Theorem 4.3.

We define a Haar measure  $d_H$  on  $H$  by  $d_H(h(a, u)) := a^{-1} da du$ , where  $du$  is the normalized Haar measure on  $U(\ell)$ . For  $\lambda_0 \in \mathcal{O}_\pm^*$  and a measurable function  $p : \mathcal{O}_\pm^* \rightarrow \mathbb{R}$ , we observe

$$\begin{aligned} \int_H p(h^{-1} \cdot \lambda_0) d_H(h) &= \int_0^{+\infty} p(a^2 \lambda_0) \frac{da}{a} = \int_{\mathcal{O}_\pm^*} p(\lambda) \frac{d\lambda}{2|\lambda|} \\ &= \int_{\mathcal{O}_\pm^*} p(\lambda) \cdot 2 \left( \frac{\pi}{2|\lambda|} \right)^{\ell+1} d\mu(\lambda). \end{aligned}$$

Thus, setting  $D_\pm(\lambda) := 2 \left( \frac{\pi}{2|\lambda|} \right)^{\ell+1}$  ( $\lambda \in \mathcal{O}_\pm^*$ ), the Duflo-Moore operator  $C_{\pm, \alpha}$  of the representation  $(L, L_{\pm, \alpha}(N))$  is given by  $\mathbf{F}[C_{\pm, \alpha}\phi](\lambda) = \sqrt{\frac{D_\pm(\lambda)}{n_\alpha}} \mathbf{F}\phi(\lambda)$  ( $\phi \in L_{+, \alpha}(N)$ ,  $\lambda \in \mathcal{O}_\pm^*$ ), where  $n_\alpha = \dim_{\mathbb{C}} \mathcal{P}_\alpha(\mathbb{C}^\ell) = \binom{\alpha+\ell-1}{\alpha}$ .

Finally, we give an example of admissible vector in  $L_{\pm, \alpha}(N)$ . We set

$$\phi_{\pm, \alpha}(n(z, c)) := \int_{\mathcal{O}_\pm^*} |\lambda| e^{-|\lambda|} \operatorname{tr} P_{\lambda, \alpha} \pi_\lambda(n(z, c))^* d\mu(\lambda).$$

Then  $(|\lambda|e^{-|\lambda|}P_{\lambda,\alpha})_{\lambda \in \mathcal{O}_\pm^*}$  belongs to both of the direct integrals  $\int_{\mathcal{O}_\pm^*}^\oplus \mathcal{B}_{\text{Tr}}(\mathcal{H}_\lambda) d\mu(\lambda)$  and  $\int_{\mathcal{O}_\pm^*}^\oplus \mathcal{B}_{\text{HS}}(\mathcal{H}_\lambda) d\mu(\lambda)$ . Applying Proposition 2.1, we see that  $\phi_{\pm,\alpha} \in L^2(N)$  and that

$$\mathbf{F}\phi_{\pm,\alpha}(\lambda) = \begin{cases} |\lambda|e^{-|\lambda|}P_{\lambda,\alpha} & (\text{a.a. } \lambda \in \mathcal{O}_\pm^*), \\ 0 & (\text{otherwise}). \end{cases}$$

Thus  $\phi_{\pm,\alpha} \in L_{\pm,\alpha}(N)$ . Furthermore we have

$$\begin{aligned} \|C_{\pm,\alpha}\phi_{\pm,\alpha}\|^2 &= n_\alpha^{-1} \int_{\mathcal{O}_\pm^*} \|\mathbf{F}\phi_{\pm,\alpha}(\lambda)\|_{\text{HS}}^2 \frac{d\lambda}{2|\lambda|} = n_\alpha^{-1} \int_{\mathcal{O}_\pm^*} n_\alpha |\lambda|^2 e^{-2|\lambda|} \frac{d\lambda}{2|\lambda|} \\ &= \frac{1}{2} \int_0^{+\infty} e^{-2\lambda} \lambda d\lambda = \frac{1}{8}, \end{aligned}$$

so that  $\phi_{\pm,\alpha}$  is admissible.

The function  $\phi_{\pm,\alpha}$  can be calculated explicitly. By definition, we have  $\phi_{-,\alpha} = \overline{\phi_{+,\alpha}}$ , so that we shall consider only  $\phi_{+,\alpha}$ . For non-negative integers  $k$  and  $\alpha$ , the Laguerre polynomial  $L_\alpha^k(s)$  is defined by  $L_\alpha^k(s) := \frac{e^s s^{-k}}{\alpha!} \left(\frac{d}{ds}\right)^\alpha [e^{-s} s^{k+\alpha}]$ . By [3, Proposition 6.2], we have

$$\text{tr } P_{\lambda,\alpha} \pi_\lambda(n(z, c))^* = e^{(ic-|z|^2/2)\lambda} L_\alpha^{\ell-1}(\lambda|z|^2).$$

Thus

$$\phi_{+,\alpha}(n(z, c)) = \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_0^{+\infty} e^{-(1-ic+|z|^2/2)\lambda} L_\alpha^{\ell-1}(\lambda|z|^2) \lambda^{\ell+1} d\lambda.$$

On the other hand, for a parameter  $|r| < 1$ , we have  $\sum_{\alpha=0}^\infty r^\alpha L_\alpha^{\ell-1}(s) = (1-r)^{-\ell} e^{-\frac{rs}{1-r}}$ . Therefore

$$\begin{aligned} \sum_{\alpha=0}^\infty r^\alpha \phi_{+,\alpha}(n(z, c)) &= \frac{2^{\ell-1}}{\pi^{\ell+1}} \int_0^{+\infty} e^{-(1-ic+|z|^2/2)\lambda} (1-r)^{-\ell} e^{-\frac{r|z|^2\lambda}{1-r}} \lambda^{\ell+1} d\lambda \\ &= \frac{2^{\ell-1}}{\pi^{\ell+1}} (1-r)^{-\ell} \cdot (\ell+1)! \left\{ (1-ic+|z|^2/2) + \frac{r|z|^2}{1-r} \right\}^{-(\ell+2)} \\ &= \frac{2^{\ell-1}(\ell+1)!}{\pi^{\ell+1}} (1-ic+|z|^2/2)^{-(\ell+2)} \\ &\quad \times (1-r)^2 \left( 1 + \frac{-1+ic+|z|^2/2}{1-ic+|z|^2/2} \cdot r \right)^{-(\ell+2)}. \end{aligned}$$

Putting  $\theta := \frac{-1+ic+|z|^2/2}{1-ic+|z|^2/2}$ , we have by the binomial theorem

$$\begin{aligned} &(1-r)^2 (1+\theta r)^{-(\ell+2)} \\ &= \sum_{\alpha=0}^\infty \frac{(-1)^\alpha (\alpha+\ell-1)!}{(\ell+1)! \alpha!} \{ \alpha(\alpha-1)\theta^{\alpha-2} + 2\alpha(\alpha+\ell)\theta^{\alpha-1} + (\alpha+\ell)(\alpha+\ell+1)\theta^\alpha \} r^\alpha. \end{aligned}$$

Hence

$$\begin{aligned} & \phi_{+,\alpha}(n(z, c)) \\ &= \frac{2^{\ell-1}}{\pi^{\ell+1}} (1 - ic + |z|^2/2)^{-(\ell+2)} \\ & \quad \times \frac{(-1)^\alpha (\alpha + \ell - 1)!}{\alpha!} \{ \alpha(\alpha - 1)\theta^{\alpha-2} + 2\alpha(\alpha + \ell)\theta^{\alpha-1} + (\alpha + \ell)(\alpha + \ell + 1)\theta^\alpha \}. \end{aligned}$$

## References

- [1] Ali, S. T., Antoine J.-P., and Gazeau, J.-P., “Coherent states, wavelets and their generalizations,” Springer-Verlag, New York, 2000.
- [2] Aniello, P., Cassinelli, G., de Vito, E., and Levrero, A., *Square integrability of induced representations of semidirect products with normal abelian subgroup*, Rev. Math. Phys., **10** (1998), 301–313.
- [3] C. Benson, J. Jenkins and G. Ratcliff, *Bounded K-spherical Functions on Heisenberg Groups*, J. Funct. Anal., **105** (1992), 409–443.
- [4] Bernier, D. and Taylor, K. F., *Wavelets from square-integrable representations*, SIAM J. Math. Anal., **27** (1996), 594–608.
- [5] Currey, B. N., *Admissibility for a class of quasiregular representations*, Canad. J. Math., **59** (2007), 917–942.
- [6] Dixmier, J., “Les  $C^*$ -algèbres et leurs Représentations,” Gauthier-Villars, Paris, 1974.
- [7] Duflo, M., and Moore, C. C., *On the regular representation of a nonunimodular locally compact group*, J. Funct. Anal., **21** (1976), 209–243.
- [8] Fabec, R., and Ólafsson, G., *The continuous wavelet transform and symmetric spaces*, Acta Appl. Math., **77** (2003), 41–69.
- [9] Folland, G., “Harmonic Analysis in Phase Space,” Princeton Univ. Press, Princeton, 1989.
- [10] Führ, H., *Wavelet frames and admissibility in higher dimensions*, J. Math. Phys., **37** (1996), 6353–6366.
- [11] Führ, H., “Abstract Harmonic Analysis of Continuous Wavelet Transforms,” Lecture Notes in Mathematics, **1863**, Springer-Verlag, New York, 2005.
- [12] Führ, H., and Mayer, M., *Continuous wavelet transforms from semidirect products: Cyclic representations and Plancherel measure*, J. Fourier Anal. Appl., **8** (2002), 375–398.
- [13] Grossmann, A., Morlet, J., and Paul, T., *Transforms associated to square integrable group representations, I and II*, J. Math. Phys., **26** (1985), 2473–2479; Ann. Inst. Henri Poincaré, **45** (1986), 293–309.
- [14] He, J., and Liu, H., *Admissible wavelets associated with the affine automorphism group of the Siegel upper half-plane*, J. Math. Anal. Appl., **208** (1997), 58–70.
- [15] Ishi, H., *Wavelet transform associated to a homogeneous Siegel domain*, Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics, 937–941, 2002.
- [16] Ishi, H., *Wavelet transforms for semidirect product groups with not necessarily commutative normal subgroups*, J. Fourier. Anal. Appl. **12** (2006), 37–52.
- [17] Kalisa, C., and Torresani, B., *n-dimensional affine Weyl-Heisenberg wavelets*, Ann. Inst. H. Poincaré Phys. Theor., **59** (1993), 201–236.
- [18] Kleppner, A., and Lipsman, R. L., *The Plancherel formula for group extensions, I and II*, Ann. Sci. Ecole Norm. Sup., **5** (1972), 459–516; *ibid.* **6** (1973), 103–132.

- [19] Liu, H., and Peng, L., *Admissible wavelets associated with the Heisenberg group*, Pacific J. Math., **180** (1997), 101–123.
- [20] Strichartz, R. S.,  *$L^p$  harmonic analysis and Radon transforms on the Heisenberg group*, J. Funct. Anal., **96** (1991), 350–406.